

**THE MAXIMAL FUNCTION AND CONDITIONAL SQUARE
FUNCTION CONTROL THE VARIATION:
AN ELEMENTARY PROOF**

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ABSTRACT. In this note we prove the following good- λ inequality, for $r > 2$, all $\lambda > 0$, $\delta \in (0, \frac{1}{2})$

$$C \cdot \nu\{V_r(f) > 3\lambda; \mathcal{M}(f) < \delta\lambda\} \leq \nu\{s(f) > \delta\lambda\} + \frac{\delta^2}{(r-2)^2} \cdot \nu\{V_r(f) > \lambda\},$$

where $\mathcal{M}(f)$ is the martingale maximal function, $s(f)$ is the conditional martingale square function, $C > 0$ is (absolute) constant. This immediately proves that $V_r(f)$ is bounded on L^p , $1 < p < \infty$ and moreover is integrable when the maximal function and the conditional square function are.

1. INTRODUCTION

Let (X, \mathcal{F}, ν) be σ -finite measure space with a filtration $(\mathcal{F}_n : n \in \mathbb{Z})$, i.e. $(\mathcal{F}_n : n \in \mathbb{Z})$ is a sequence of σ -fields such that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \mathcal{F}$. For a martingale $(f_n : n \in \mathbb{Z})$ we define the maximal function, the square function

$$\mathcal{M}(f) = \sup_{n \in \mathbb{Z}} |f_n|, \quad S(f) = \left(\sum_{n \in \mathbb{Z}} |f_n - f_{n-1}|^2 \right)^{1/2},$$

and a conditional square function

$$s(f) = \left(\sum_{n \in \mathbb{Z}} \mathbb{E}[|f_n - f_{n-1}|^2 | \mathcal{F}_{n-1}] \right)^{1/2}.$$

For a dyadic filtration on \mathbb{R}^d we have $S(f) = s(f)$ since the square of a martingale difference $|f_n - f_{n-1}|^2$ is \mathcal{F}_{n-1} -measurable. It is well known (see [2, Theorem 9] and [4, Theorem 1], see also [3, Theorem 1.1]) that for each $p \in [1, \infty)$ there exists $C_p > 0$ such that

$$(1.1) \quad C_p^{-1} \|\mathcal{M}(f)\|_{L^p(\nu)} \leq \|S(f)\|_{L^p(\nu)} \leq C_p \|\mathcal{M}(f)\|_{L^p(\nu)}.$$

Also, by convexity Lemma (see [3, Theorem 3.2]) for each $p \in [2, \infty)$ there is $C_p > 0$ satisfying

$$\|s(f)\|_{L^p(\nu)} \leq C_p \|S(f)\|_{L^p(\nu)}.$$

For $p \in (0, 2)$, we have

$$\|\mathcal{M}f\|_{L^p(\nu)} \leq C_p \|s(f)\|_{L^p(\nu)}.$$

Date: October 30, 2014.

2010 Mathematics Subject Classification. Primary 60G42, 60E15; Secondary 47B38, 46N30.

In general, the conditional square function s is not necessary bounded on $L^p(\Omega, \nu)$, for $p \in (1, 2)$. Let us recall that the filtration $(\mathcal{F}_n : n \in \mathbb{Z})$ is regular if there is $R \geq 1$ such that for all nonnegative martingales $(g_n : n \in \mathbb{Z})$,

$$g_n \leq Rg_{n-1}.$$

If the filtration $(\mathcal{F}_n : n \in \mathbb{Z})$ is regular then (1.1) is valid for all $p \in (0, \infty)$; moreover, for all $p \in (0, \infty)$ there is $C_p > 0$ such that

$$C_p^{-1} \|s(f)\|_{L^p(\nu)} \leq \|S(f)\|_{L^p(\nu)} \leq C_p \|s(f)\|_{L^p(\nu)}.$$

Another family of operators which measure oscillation are the r -variation operators defined for $r \geq 1$ by

$$V_r(f) = \sup_{n_0 < n_1 < \dots < n_J} \left(\sum_{j=1}^J |f_{n_j} - f_{n_{j-1}}|^r \right)^{1/r}.$$

These variation operators are more difficult to control than the maximal function \mathcal{M} . In fact, for any $n_0 \in \mathbb{Z}$, one may pointwise dominate

$$\mathcal{M}(f) \leq V_r(f) + |f_{n_0}|,$$

where $r \geq 1$ is arbitrary. We further remark that the variation operators become larger (more sensitive to oscillation) as r decreases. The fundamental boundedness result concerning the r -variation operators is due to Lépingle.

Theorem 1.1 ([7]). *For each $p \in [1, \infty)$ there is $A_p > 0$ such that for all $f \in L^p(X, \nu)$ and $r > 2$*

$$\|V_r(f)\|_{L^p(\nu)} \leq A_p \frac{r}{r-2} \|f\|_{L^p(\nu)}, \quad (p > 1),$$

and for all $\lambda > 0$

$$\nu\{V_r(f) > \lambda\} \leq A_1 \frac{r}{r-2} \lambda^{-1} \|f\|_{L^1(\nu)}, \quad (p = 1).$$

We remark that the range of r in the above theorem is sharp, since these estimates can fail for $r \leq 2$, (see e.g. [5, 9]).

By now, comparatively simple proofs of Lépingle's theorem can be found in Pisier and Xu [8] and Bourgain [1] (see also [6]). The idea was to leverage known estimates for jump inequalities to recover variational estimates. Let us recall that the number of λ -jumps, denoted by $N_\lambda(f)$, is equal to the supremum over $J \in \mathbb{N}$ such that there is an increasing sequence $n_0 < n_1 < \dots < n_J$ satisfying

$$|f_{n_j} - f_{n_{j-1}}| > \lambda$$

for all $1 \leq j \leq J$. The key result concerning λ -jumps is the following theorem.

Theorem 1.2 ([1, 8]). *For each $p \in [1, \infty)$ there exist $B_p > 0$ such that for all $f \in L^p(X, \nu)$ and $\lambda > 0$*

$$\|\lambda N_\lambda^{1/2}(f)\|_{L^p(\nu)} \leq B_p \|f\|_{L^p(\nu)}, \quad (p > 1),$$

and

$$\nu\{\lambda N_\lambda^{1/2}(f) > t\} \leq B_1 t^{-1} \|f\|_{L^1(\nu)}, \quad (p = 1),$$

for any $t > 0$.

The goal of this note is to provide a new and elementary proof of Lépingle's result. The significance of our approach is that it sheds new insight into the relationship between maximal function, conditional square function, and variation operator. Specifically, we prove the following theorem.

Theorem A. *There is $C > 0$ such that for all $\delta \in (0, \frac{1}{2})$, $r > 2$ and $\lambda > 0$*

$$(1.2) \quad C \cdot \nu\{V_r(f) > 3\lambda; \mathcal{M}(f) \leq \delta\lambda\} \leq \nu\{s(f) > \delta\lambda\} + \frac{\delta^2}{(r-2)^2} \cdot \nu\{V_r(f) > \lambda\}.$$

In particular, by integrating distribution functions we obtain that for all $p \in (0, \infty)$ and $r > 2$, $V_r(f) \in L^p(X, \nu)$ whenever $s(f)$ and $\mathcal{M}(f)$ are in $L^p(X, \nu)$.

1.1. Acknowledgments. The authors wish to thank Konrad Kolesko, Christoph Thiele, and Jim Wright for helpful discussions, and Michael Lacey and Terence Tao for their careful reading and support. Furthermore, this work was completed during the “Harmonic Analysis and Partial Differential Equations” trimester program at Hausdorff Research Institute for Mathematics; the authors wish to thank HIM for their hospitality.

2. THE PROOF

We begin with a preliminary lemma.

Lemma 2.1. *There is $C > 0$ such that for any $A \in \mathcal{F}_m$, all $\lambda > 0$, and $\delta \in (0, \frac{1}{2})$*

$$\nu\{A; V_r(f) > \lambda; \mathcal{M}(f) \leq \delta\lambda\} \leq C\lambda^{-2}(r-2)^{-2} \int_A |f|^2 d\nu$$

for each $f \in L^2(X, \nu)$ satisfying

$$f_n \cdot \mathbf{1}_A = 0$$

for all $n \leq m$.

Proof. By homogeneity, it suffices to prove the result with $\lambda = 1$. We can pointwise dominate the variation as in [1, §3]

$$(V_r(f))^r \leq \sum_{l \in \mathbb{Z}} 2^{rl} N_{2^l}(f).$$

Let $s = (r+2)/2$. Since $\mathcal{M}(f) < \delta < 1/2$, the above sum runs over $l \leq 0$, which leads to the containment

$$\{A; V_r(f) > 1; \mathcal{M}(f) < \delta\} \subset \left\{A; \sum_{l \leq 0} 2^{rl} N_{2^l}(f) > 1\right\} \subset \bigcup_{l \leq 0} \{2^{sl} N_{2^l}(g) > c_r\},$$

where $g = f \cdot \mathbf{1}_A$ and

$$c_r^{-1} = \frac{1}{2} \sum_{l \leq 0} 2^{(r-2)l/2}.$$

Let us observe that $c_r = o(r-2)$. In light of Theorem 1.2, this immediately leads to the majorization¹

$$\nu\{A; V_r(f) > 1; \mathcal{M}(f) < \delta\} \lesssim c_r^{-1} \sum_{l \leq 0} 2^{(s-2)l} \int_A |f|^2 d\nu \lesssim (r-2)^{-2} \int_A |f|^2 d\nu.$$

□

¹We write $X \lesssim Y$, or $Y \gtrsim X$ to denote the estimate $X \leq CY$ for an absolute constant $C > 0$.

Proof of Theorem A. By homogeneity, it will suffice to prove (1.2) for $\lambda = 1$.

Let $B = \{s(f) > \delta\}$, $B^* = \{\mathcal{M}(\mathbb{1}_B) > 1/2\}$ and $G = (B^*)^c$. By Doob's inequality, we have

$$\nu(B^*) \lesssim \int |\mathbb{1}_B|^2 d\nu = \nu\{s(f) > \delta\}.$$

Therefore, it is enough to show that there is $C > 0$ such that for all $\delta \in (0, \frac{1}{2})$ and any $N \in \mathbb{N}$

$$\nu\{V_r(f_n : -N \leq n) > 3; \mathcal{M}(f) < \delta; G\} \leq C\delta^2 \cdot \nu\{V_r(f) > 1\}.$$

Let σ be a stopping time defined to be equal to the minimal $m \in \mathbb{Z}$ such that

$$V_r(f_n : -N \leq n \leq m) > 1.$$

Notice, that on the set $\{V_r(f_n : -N \leq n) > 3\}$, we have $-N \leq \sigma < \infty$. Next,

$$V_r(f_n : -N \leq n) \leq V_r(f_n - f_{n \wedge \sigma} : -N \leq n) + 2\mathcal{M}(f) + V_r(f_{n \wedge \sigma} : -N \leq n < \sigma),$$

thus for $g = f - f_\sigma$ we have

$$\{V_r(f_n : -N \leq n) > 3; \mathcal{M}(f) < \delta; G\} \subseteq \{V_r(g) > 1; \mathcal{M}(g) < 2\delta; G\}.$$

We are going to prove that for each $m \in \mathbb{Z}$

$$\nu\{V_r(g) > 1; \mathcal{M}(g) < 2\delta; G; \sigma = m\} \lesssim \delta^2 \cdot \nu\{\sigma = m\}.$$

For $n \in \mathbb{Z}$ we define $U_n = \{x : \mathbb{E}[\mathbb{1}_B | \mathcal{F}_n](x) < 1/2\}$. We notice that, if $x \in G$ then $x \in U_n$ for all $n \in \mathbb{Z}$. Let

$$\tilde{g}(x) = \sum_{n \in \mathbb{Z}} (g_n(x) - g_{n-1}(x)) \cdot \mathbb{1}_{U_{n-1}}(x).$$

We observe that $g_k(x) = \tilde{g}_k(x)$ for all $x \in G$ and $k \in \mathbb{Z}$. Indeed, $(g_n - g_{n-1}) \cdot \mathbb{1}_{U_{n-1}}$ is \mathcal{F}_n -measurable and

$$\mathbb{E}[(g_n - g_{n-1}) \cdot \mathbb{1}_{U_{n-1}} | \mathcal{F}_{n-1}] = 0.$$

Thus for $x \in G$ we have

$$\tilde{g}_k(x) = \sum_{n \leq k} (g_n(x) - g_{n-1}(x)) \cdot \mathbb{1}_{U_{n-1}}(x) = g_k(x).$$

Therefore, we obtain

$$\begin{aligned} \nu\{V_r(g) > 1; \mathcal{M}(g) < 2\delta; G; \sigma = m\} &= \nu\{V_r(\tilde{g}) > 1; \mathcal{M}(\tilde{g}) < 2\delta; G; \sigma = m\} \\ &\leq \nu\{V_r(\tilde{g}) > 1; \mathcal{M}(\tilde{g}) < 2\delta; \sigma = m\}. \end{aligned}$$

Because $\tilde{g} = 0$ on the set $\{\sigma = m\}$, by Lemma 2.1 we conclude

$$\nu\{V_r(\tilde{g}) > 1; \mathcal{M}(\tilde{g}) < 2\delta; \sigma = m\} \lesssim (r-2)^{-2} \int_{\{\sigma=m\}} |\tilde{g}|^2 d\nu.$$

Next, since s preserves L^2 -norm thus

$$\int_{\{\sigma=m\}} |\tilde{g}|^2 d\nu = \int s(\tilde{g} \cdot \mathbb{1}_{\{\sigma=m\}})^2 d\nu = \sum_{n \in \mathbb{Z}} \int_{\{\sigma=m\}} \mathbb{E}[|g_n - g_{n-1}|^2 | \mathcal{F}_{n-1}] \cdot \mathbb{1}_{U_{n-1}} d\nu.$$

Since $\mathbf{1}_{U_{n-1}} \leq 2 \cdot \mathbb{E}[\mathbf{1}_{B^c} | \mathcal{F}_{n-1}]$ we get

$$\begin{aligned} \int_{\{\sigma=m\}} |\tilde{g}|^2 d\nu &\leq 2 \sum_{n \in \mathbb{Z}} \int_{\{\sigma=m\}} \mathbb{E}[|g_n - g_{n-1}|^2 | \mathcal{F}_{n-1}] \cdot \mathbb{E}[\mathbf{1}_{B^c} | \mathcal{F}_{n-1}] d\nu \\ &= 2 \int_{\{\sigma=m\}} s(f)^2 \cdot \mathbf{1}_{B^c} d\nu \end{aligned}$$

which is bounded by $2\delta^2 \cdot \nu\{\sigma = m\}$. \square

3. APPLICATIONS TO DYADIC A_∞ -WEIGHTS

We remark that in the case of the dyadic filtration on \mathbb{R}^d , the proof generalizes to handle measures given by w , dyadic A_∞ -weights. First, let us recall the following definition.

Definition 3.1. A non-negative locally integrable function w belongs to *dyadic* A_∞ , if for every $\epsilon > 0$ there exists $\gamma > 0$ so that for every dyadic interval I and any measurable set $E \subset I$, if $|E| \leq \gamma \cdot |I|$ then

$$(3.1) \quad w(E) \leq \epsilon w(I).$$

If additionally, there is $C > 0$ such that for all dyadic intervals I

$$C^{-1}w(I_l) \leq w(I_r) \leq Cw(I_l),$$

where I_l and I_r are, respectively, left and right children of I , then w is called dyadic doubling.

Corollary 1. *Let w be a dyadic A_∞ -weight. There exist $C > 0$ so that for each $\epsilon > 0$ and $r > 2$ there is $\delta > 0$ such that for all $\lambda > 0$*

$$w\{V_r(f) > 3\lambda; \mathcal{M}(f) < \delta\lambda\} \leq C \cdot w\{S(f) > \delta\lambda\} + \epsilon \cdot w\{V_r(f) > \lambda\}.$$

Proof. Using the notation as in the proof of Theorem A we may write

$$\{V_r(f) > 3; \mathcal{M}(f) < \delta; G; \sigma = m\} \subseteq \{\sigma = m\}$$

and

$$|\{V_r(f) > 3; \mathcal{M}(f) < \delta; G; \sigma = m\}| \leq C \frac{\delta^2}{(r-2)^2} \cdot |\{\sigma = m\}|.$$

Given $\epsilon > 0$ we take $\delta > 0$ small enough so that $C \frac{\delta^2}{(r-2)^2} \leq \gamma$. Then, by (3.1) we get

$$w\{V_r(f) > 3; \mathcal{M}(f) < \delta; G; \sigma = m\} \leq \epsilon \cdot w\{\sigma = m\}.$$

Since for the dyadic filtration $S(f) = s(f)$ we conclude the proof. \square

Again, by integrating distribution functions for each $p \in [1, \infty)$ and $r > 2$ we can find $C_{p,r} > 0$ such that

$$C_{p,r} \|V_r(f)\|_{L^p(w)} \leq \|S(f)\|_{L^p(w)} + \|\mathcal{M}(f)\|_{L^p(w)}.$$

By [10, §2], for each w , a dyadic A_∞ -weight there is $C_p > 0$

$$\|\mathcal{M}(f)\|_{L^p(w)} \leq C_p \|S(f)\|_{L^p(w)},$$

so the square function alone dominates V_r in $L^p(w)$. In the case where w is a dyadic doubling, we have the reverse inequality as well

$$\|S(f)\|_{L^p(w)} \leq C_p^{-1} \|\mathcal{M}(f)\|_{L^p(w)},$$

and thus the maximal function alone dominates V_r in $L^p(w)$.

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